OPEN CHALLENGE ’21 Solutions

WET NELLIE

Let the tension in the rope be T_0 = 1 tonne wt = 1000 kg wt
Each complete turn around the tree reduces the pull by 70%
$\therefore T_1 = 0.3T_0 = 300$ kg wt
$T_2 = 0.3T_1 = 90$ kg wt
$T_3 = 0.3T_2 = 27$ kg wt
As the lady can support 50 kg wt, the elephant can be held with three, but not two, turns.

Let 0.5 turn reduce the pull by a factor a
Then $T_3 = aT_{2.5} = a^2T_2$
$\therefore a^2 = 0.3$ and $a = 0.5477$
$\therefore T_{2.5} = aT_2 = 0.5477\text{x}90$
$= 49.295$ kg wt
The elephant can be held with 2.5 turns (just!)

Alternative method using logs

Let the number of turns round the tree be n
$1000 \times 0.3^n \leqslant 50$
$0.3^n \leqslant 0.05$
$n \geqslant \log_{0.3} 0.05$
$n \geqslant 2.488\ldots$
Therefore the elephant can be held with 3 complete turns.

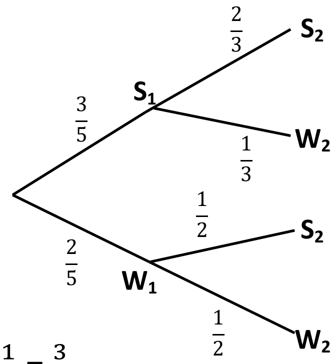
As 2.5⩾2.488 the elephant can be held with half a turn less.

SOME SUNNY DAY

Let P(S) and P(W) be the probabilities of sunny and wet days during Basil and Rosemary’s visits.

$P(S) = \frac{2}{3}P(S) + \frac{1}{2}P(W)$
$\therefore \frac{1}{3}P(S) = \frac{1}{2}P(W)$
$\therefore P(S) = \frac{3}{2}P(W)$
Days are either wet or sunny
$P(S) + P(W) = 1$
$\therefore P(W) = \frac{2}{5}$ and $P(S) = \frac{3}{5}$
Therefore in 10 visits of 4 days each the number of wet days = $\frac{2}{5} \times 40 = 16$
An estimated 16 days have been wet in their last ten visits.

On this visit, the second day is sunny.



Using conditional probability
$P(S_1|S_2) = P(S_1 \cap S_2) / P(S_2)$

$P(S_1 \cap S_2) = \frac{3}{5} \times \frac{2}{3} = \frac{2}{5}$

$P(S_2) = P(S_1) \times P(S_2) + P(W_1) \times P(S_2) = \frac{3}{5} \times \frac{2}{3} + \frac{2}{5} \times \frac{1}{2} = \frac{3}{5}$

$P(S_1|S_2) = \frac{2}{5} / \frac{3}{5} = \frac{2}{3}$

Therefore the probability that the previous day was described as fine is 2/3.

There are other ways of reaching these solutions.

HOT ICE

As the mass of water was given this is constant whether as ice or liquid.
Mass of water = 3×10^{19}kg
Take the density of water as 1000kgm^{-3}
Volume = Mass/Density
Volume of water = $3 \times 10^{19}/1000 = 3 \times 10^{16}\ m^3 = 0.00003 \times 10^{21}\ m^3$
It was assumed that the earth was a sphere and its mean radius was approximately 6371km = 6.371×10^6 m
Volume of sphere = $\frac{4}{3}\pi r^3$
= $1.0832069 \times 10^{21}\ m^3$

New volume =$1.0832069 \times 10^{21} + 0.00003 \times 10^{21}\ m^3$
=$1.0832369 \times 10^{21}\ m^3$
New radius = $\sqrt[3]{\frac{3 \times 1.083269 \times 10^{21}}{4\pi}}$
= 6371058.8156 m
Therefore the radius is increased by **58.8156 m = depth of water.**

Surface area of earth = $4\pi r^2 = 4\pi(6.371 \times 10^6)^2 = 5.10064 \times 10^{14}\ m^2$
For a flat disc, depth of water = $3 \times 10^{16}/5.10064 \times 10^{14}$
Depth of water = **58.8161 m**

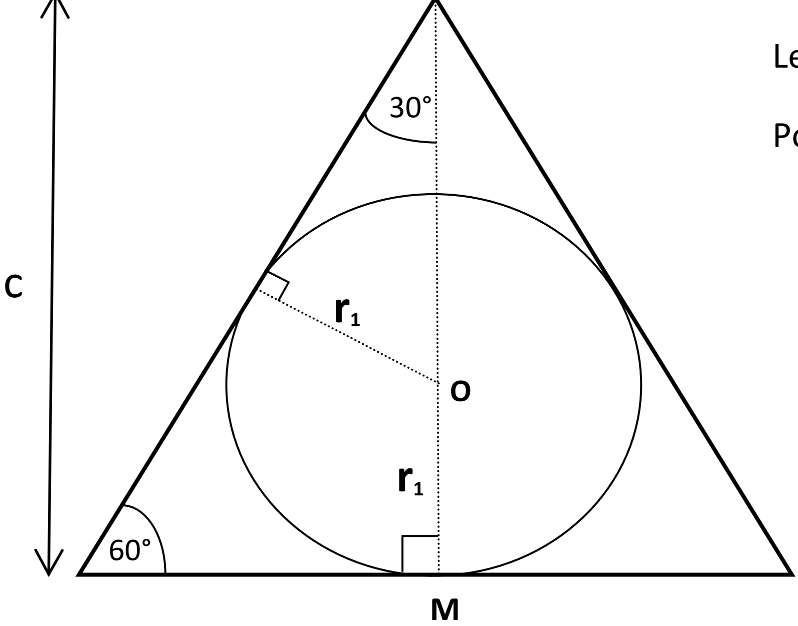
The greater depth on the disc is because the water is modelled as a cylinder with constant radius. The water on the surface of the sphere forms a shell with the outer radius being greater than the inner radius.
Using the approximation 6371km for the radius of the earth the depth differs by only half a millimetre. This is very small as the volume of water is considerably less than the volume of the earth.

DIPPY SHEILA

Sheila advances 4 each time (1 dipped and 3 jumped).
As Sheila is number 85 she has 84 sheep before her.
84/4 = 21 sheep dipped and 63 sheep overtaken.
Therefore 21 sheep are dipped before Sheila.

This year Sheila still makes 21 jumps but this time she advances 5 each time (2 dipped and 3 jumped).
21x5 = 105 and so there are 105 sheep in front of Sheila and she is number 106.
But as two are dipped simultaneously, Sheila could be the second of the two waiting sheep and this would make her number 107.
However, there could be fewer than three sheep waiting at the end. Sheila would then only have to overtake 0, 1 or 2 sheep.
So Sheila could be number 103, 104 or 105.
This means that Farmer Giles could have 103, 104, 105, 106 or 107 sheep.

END OF HIS TETHER

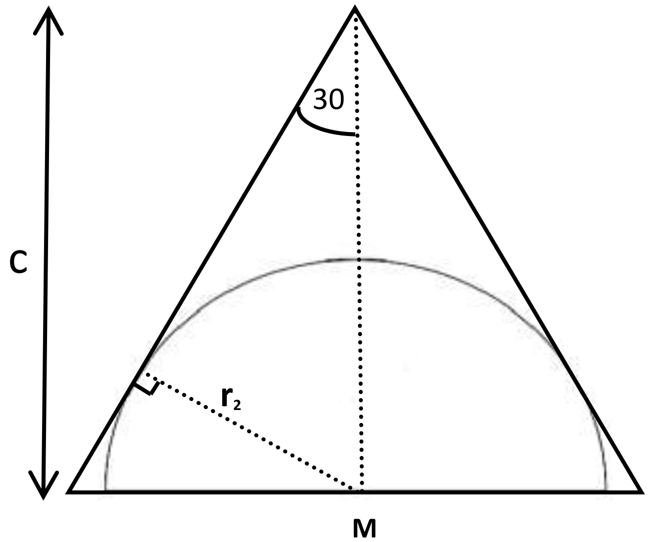


Let the length of the median be c

Post at centre of field

$r_1 = \frac{1}{3}c$ (Intersection of medians)

Area of circle $= \pi(\frac{1}{3}c)^2$
$= 1/9\ \pi c^2$ ($\approx 0.349c^2$)



Post at midpoint of wall

$r_2 = \frac{1}{2}c$

Area of semi-circle $= \frac{1}{2}\pi(\frac{1}{2}c)^2$
$= 1/8\ \pi c^2$ ($\approx 0.393c^2$)

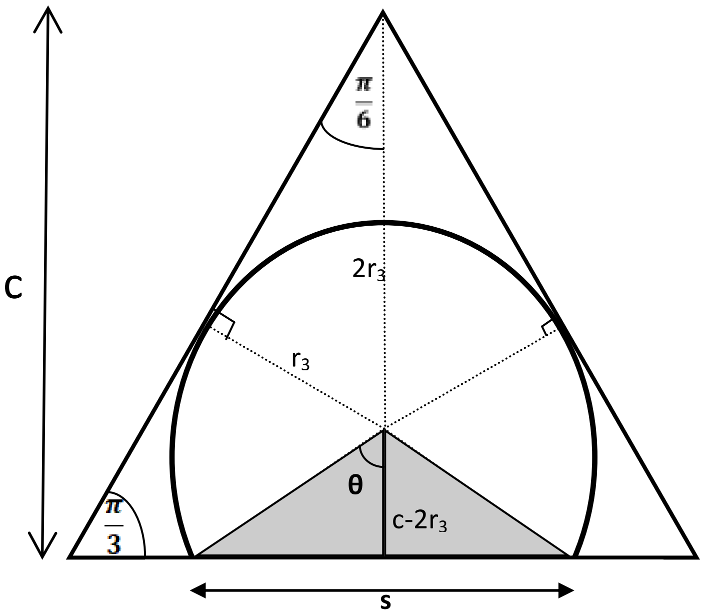
$1/8\ \pi c^2 > 1/9\ \pi c^2$

Therefore the post at the midpoint of the wall allows the goat to eat more grass than the post at the centre of the field.

The length of the tether for the area available to the goat must be just short of the radius in each case as the goat must not eat the hedge.

<u>Extension</u>

In fact, if the post is placed at any point on the line OM the goat can eat more grass than when the post is at O. In this case the area is a segment of a circle.



Distance from apex to centre = $2r_3$

$\cos\theta = \frac{c - 2r_3}{r_3} \Rightarrow r_3 = \frac{c}{(2+\cos\theta)}$

Area segment = area sector + area shaded triangle

$A = \frac{1}{2} r_3^2(2\pi - 2\theta) + \frac{1}{2} r_3^2 \sin 2\theta$

$A = r_3^2(\pi - \theta + \frac{1}{2}\sin 2\theta)$

$A = \frac{(\pi - \theta + \frac{1}{2}\sin 2\theta)\, c^2}{(2+\cos\theta)^2}$

The value of θ for maximum area can now be found by calculus or by using graph drawing software.

<u>Using calculus</u>

For a maximum or minimum, $\frac{dA}{d\theta} = 0 \quad \Rightarrow \quad 2\sin\theta(\pi - \theta - 2\sin\theta) = 0$

$\sin\theta = 0$ (min) or $\sin\theta = \frac{1}{2}(\pi - \theta)$ (max)

$\sin\theta = \frac{1}{2}(\pi - \theta)$ can be solved by iteration or by finding the intersection of the graphs of $\sin\theta$ and $\frac{1}{2}(\pi - \theta)$

For maximum area, $\theta \approx 1.246$ radians $\approx 71.4°$

$r_3 \approx 0.431c$

$\mathbf{A_{max} \approx 0.409\, c^2}$

For maximum area the post must be placed 0.138c along the median from the wall.

<u>Interesting fact</u>

Maximum area is when the length of the arc of the segment = twice the length of the chord (marked s in diagram).

A WAY TO WEIGH HAY

Let the weights of the bales in pounds in increasing size be A, B, C, D, E.
Therefore A+B=110lb, A+C=112lb, C+E = 120lb, D+E=121lb (1)
Sum of all weighings = 4(A+B+C+D+E) = 1156

(A+B)+C+(D+E) = 289

110+C+121 = 289

$\therefore$ C = 58

From (1), **weight of each of the bales = 54lb, 56lb, 58lb, 59lb, 62lb.**

FAIR SHARES

Proceeds from the sale of n sheep = £n^2
At the final share out,
A has £10, B has £10, C has £10, D has £x (x <10 and x last digit of a square number)

Each of A, B and C gives £y to D $10 - y = x + 3y$ $x = 10 - 4y$ $\therefore y = 1$ or 2 (x, y integers) $\therefore x = 6$ or 2 No square number has units digit 2 $\therefore x = 6$ and $y = 1$	*Alternative method using Modular Arithmetic* $n^2 \equiv (30 + x)$mod 40 Quadratic residues mod 40 are 1, 4, 9, 20, 24, 25, **36** $\therefore x = 6$

Each of A, B and C gives £1 to D (and all have £39).

There are many possible values for the number of sheep in the flock.
These can be given by the formula 20m $\pm$ 6 (m a natural number)