OPEN CHALLENGE '14
SOLUTIONS

1. EIRIK'S SHIELD

Let x be the area of the final piece. Then the total area is $32x$.

Now $\Sigma p = 32x$, where p is a prime number.

The only value of Σp less than 1000 that has a factor of 32 is 160 (i.e. $p = 2, 3, 5, \ldots, 31$)

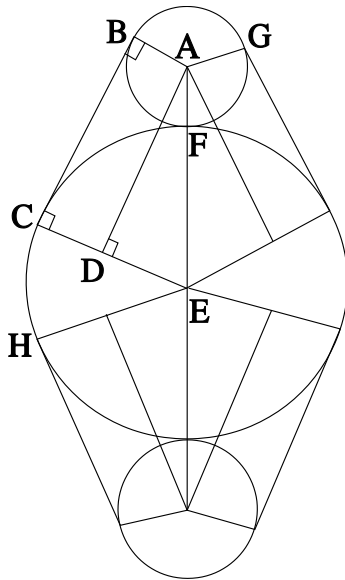This gives the area of the last piece as 5 units.

The area that fell to the ground on the first blow was 80 units (7, 19, 23 and 31)

The area that fell to the ground on the second blow was 40 units (11 and 29)

The area that fell to the ground on the third blow was 20 units (3 and 17)

The area that fell to the ground on the fourth blow was 15 units (2 and 13)

2. A WORK OF ART

As the band has a width of 1cm there are 3 options for its length.

The internal length – using radii 4cm and 9cm.

The external length – using radii 5cm and 10cm.

The midway length – using radii 4.5cm and 9.5cm.

Using the first case

△ADE is a 5,12,13 right angled triangle

∴AD = 12cm ⇒ BC = 12cm

$\cos\angle AED = 5/13$ ⇒ $\angle CEH = 180° - 2\cos^{-1}(5/13) = 45.24°$

Length of arc HC = $2\pi r \times \theta/360$ ⇒HC = 7.106 cm

As BA is parallel to CE

$\angle BAG = 2 \times \angle AED = 134.76°$

Length of arc BG = $2\pi r \times \theta/360$ ⇒BG= 9.408cm

Now the length of the band is made up of

$4 \times BC + 2 \times arcHC + 2 \times arcBG = 81.03cm = 81cm$ to nearest cm

Similarly the external length is 87.31cm = 87cm to nearest cm

and the midway length is 84.17cm = 84cm to nearest cm

3. KNUTSON'S HOARD

Let the number of gold coins be x

It can be shown that the proportion each son obtains is:

$$A = \frac{409}{3840}x \quad B = \frac{181}{768}x \quad C = \frac{1219}{13440}x \quad D = \frac{5147}{13440}x \quad E = \frac{165}{896}x$$

The smallest number of coins is the LCM of the denominators.

$768=2^8 \times 3 \quad 896=2^7 \times 7 \quad 3840=2^8 \times 3 \times 5 \quad 13440=2^7 \times 3 \times 5 \times 7$

Thus the LCM is $2^8 \times 3 \times 5 \times 7 = 26\,880$ coins.

Thus the smallest number of coins in the hoard was 26 880.

4. BEST OF THREE

Eirik can ensure he wins by going second and placing his rook, at the start and on every subsequent move, on the same diagonal as his son's rook. He can then force Bram's rook into a corner and win.

In the case of 15 stones, the first player wins if he starts by taking 2 stones. Then when he holds an odd number and leaves 1, 8 or 9 he wins, and when he holds an even number and leaves 4, 5 or 12 he also wins. He can always do one or other of these things until the end of the game, and so defeat his opponent.

In the case of 13 stones the first player must lose if his opponent plays correctly.

The only numbers that the first player should lose are 5 stones and multiples of 8 added to 5. (i.e. $5 + 8n$ where $n \in \mathbb{N}$).

You cannot start with an even number of stones as both players will either end with an odd number of stones (hence both win) or both end with an even number of stones (and no one wins).

5. HOW HIGH CAN YOU GO?
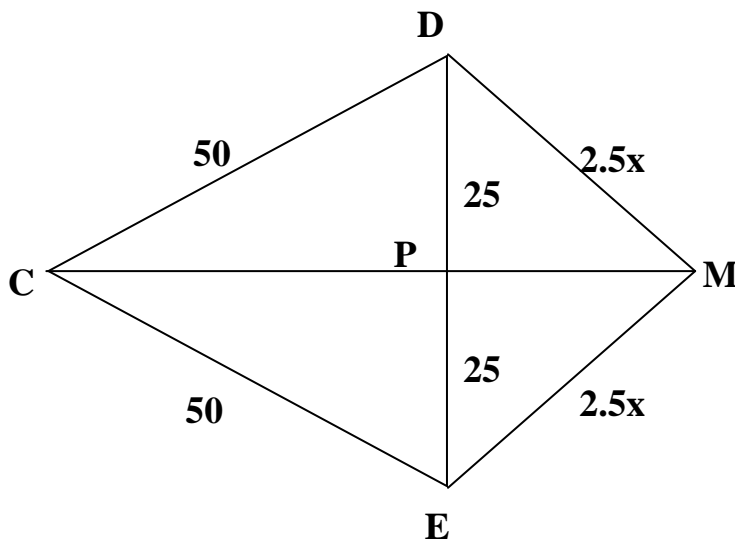
As CDME is a kite CM bisects DE at right angles

Let the height of the mast be x

In right angled $\triangle$CPD

$CP^2 = CD^2 - PD^2$

$\therefore CP = 25\sqrt{3} \quad \Rightarrow PM = 5x - 25\sqrt{3}$

In right angled $\triangle$MPD

$(2.5x)^2 = 25^2 + (5x - 25\sqrt{3})^2$

$3x^2 - 40\sqrt{3}x + 400 = 0$

Solving this gives

$x = \frac{20\sqrt{3}}{3} = 11.55\text{m}$

Thus the height of the mast is about 11.5 m

6. CROSSING THE SEA

Distance = Speed × Time

Hence the first 15nautical miles took 1.25 hours=75minutes

i.e. a time of 5minutes for each nautical mile.

The next nautical mile took 5×1.2minutes=6minutes.

Hence the time for the first 16nautical miles is 81minutes (1hour 21minutes).

The whole crossing took $75 + \frac{6(1 - 1.2^{10})}{1 - 1.2}$ minutes $= 230.75$ minutes.

Thus the whole crossing took 231 minutes (3hours 51minutes)